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LINEAR DISCREPANCY OF TOTALLY UNIMODULAR MATRICES*†

BENJAMIN DOERR

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We show that the linear discrepancy of a totally unimodular $m \times n$ matrix A is at most

$$\mathrm{lindisc}(A) \leq 1 - \tfrac{1}{n+1}.$$

This bound is sharp. In particular, this result proves Spencer's conjecture $\mathrm{lindisc}(A) \leq (1 - \frac{1}{n+1})\,\mathrm{herdisc}(A)$ in the special case of totally unimodular matrices. If $m \geq 2$, we also show $\mathrm{lindisc}(A) \leq 1 - \frac{1}{m}$.

Finally we give a characterization of those totally unimodular matrices which have linear discrepancy $1 - \frac{1}{n+1}$: Besides $m \times 1$ matrices containing a single non-zero entry, they are exactly the ones which contain $n+1$ rows such that each n thereof are linearly independent. A central proof idea is the use of linear programs.

1. Introduction and Results

Let $A \in \mathbb{R}^{m \times n}$ be any real matrix and $p \in [0,1]^n$. The *linear discrepancy* of A with respect to p is defined by

$$\mathrm{lindisc}(A,p) := \min_{z \in \{0,1\}^n} \|A(p-z)\|_\infty,$$

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† A similar result has been independently obtained by T. Bohman and R. Holzman and presented at the Conference on Hypergraphs (Gyula O. H. Katona is 60), Budapest, in June 2001.

the linear discrepancy of A is

$$\operatorname{lindisc}(A) := \max_{p\in[0,1]^n} \operatorname{lindisc}(A,p).$$

The linear discrepancy can be seen as a generalized notion of discrepancy as well as a measure of how well a solution of a linear system of equations $Ax = b$ can be rounded to an integer one. The latter is also called *lattice approximation problem*. Let us briefly sketch the two aspects.

The *discrepancy* of A is

$$\operatorname{disc}(A) := \min_{\chi\in\{-1,1\}^n} \|A\chi\|_\infty.$$

Discrepancy is a measure of how well the columns of A can be partitioned into two classes such that for each row the two sums of its entries in each class are nearly equal. If A is the incidence matrix of a hypergraph this corresponds to the problem of coloring the vertex set with two colors in such a way that all hyperedges are colored in a balanced manner. We refer to Beck and Sós [1] for a more extensive treatment of combinatorial discrepancies. Immediately we see $\operatorname{disc}(A)=2\operatorname{lindisc}\big(A,\frac{1}{2}\mathbf{1}_n\big)\le 2\operatorname{lindisc}(A)$.[1] Therefore the linear discrepancy is a generalization of the notion of discrepancy, where we have weights assigned to the columns describing the ratio in which (in average) we would like this column to belong to each of the two partition classes. Discrepancies also yield an upper bound for the linear discrepancy as shown by Lovász, Spencer and Vesztergombi [10].

The following elementary remark connects linear discrepancies with the problem of approximate integer solutions of linear systems.

Remark. Let $A\in\mathbb{R}^{m\times n}$ and $b\in\mathbb{R}^m$ such that the linear system $Ax=b$ has a solution x. Let $p\in[0,1]^n$ such that $p-x\in\mathbb{Z}^n$. Then there is a $z\in\mathbb{Z}^n$ such that $\|x-z\|_\infty\le 1$ and $\|Az-b\|_\infty\le\operatorname{lindisc}(A,p)$.

An $m\times n$ matrix A is called *totally unimodular* if each square submatrix has determinant $-1, 0$ or 1. In particular, $A\in\{-1,0,1\}^{m\times n}$. Totally unimodular matrices arise naturally in several areas of mathematics. For example, the vertex-edge incidence matrix of an undirected graph is totally unimodular if and only if the graph is bipartite. More general, the $(-1,0,1)$ vertex-arc incidence matrix of a directed graph is totally unimodular. This fact can be used to prove the max-flow min-cut theorem of Ford and Fulkerson [5].

[1] Sometimes the linear discrepancy is defined by $\max_{p\in[-1,1]^n}\min_{\chi\in\{-1,1\}^n}\|A(p-\chi)\|_\infty$. This makes the linear discrepancy a direct generalization of the discrepancy, but puts less emphasis on the connection to the rounding problem. This notion is larger than the one we use by a factor of 2.

The discrepancy problem for totally unimodular matrices is well-understood. Their discrepancy is at most one. By definition, submatrices of totally unimodular matrices are totally unimodular, hence the discrepancy of all submatrices is at most one as well. The beautiful theorem of Ghouila-Houri [6] states that also the converse holds:

Theorem (Ghouila-Houri, 1962). *A matrix is totally unimodular if and only if each submatrix has discrepancy at most one.*

Defining the *hereditary discrepancy* herdisc$(\cdot)$ of a matrix to be the maximal discrepancy among all submatrices, we see that totally unimodular matrices are exactly the ones having hereditary discrepancy at most one. Contrary to the discrepancy and the hereditary discrepancy, for the linear discrepancy of totally unimodular matrices a sharp upper bound was missing so far.

Previous Results

Lovász, Spencer and Vesztergombi [10] show

$$\operatorname{lindisc}(A) \leq \operatorname{herdisc}(A)$$

for any matrix A. Thus we immediately have $\operatorname{lindisc}(A) \leq 1$ for a totally unimodular matrix A. Spencer conjectures that even $\operatorname{lindisc}(A) \leq (1 - \frac{1}{n+1}) \operatorname{herdisc}(A)$ holds for any A. This would immediately yield $\operatorname{lindisc}(A) \leq 1 - \frac{1}{n+1}$, but this over 15 years old conjecture seems far from being proven. The current best bound is $\operatorname{lindisc}(A) \leq (1 - \frac{1}{2m}) \operatorname{herdisc}(A)$, cf. [3]. For totally unimodular matrices we conclude $\operatorname{lindisc}(A) \leq 1 - \frac{1}{n(n+1)}$. This follows from Theorem 4.2 of [7], which implies $m \leq \frac{1}{2}(n+1)n$ for a totally unimodular matrix having the property that any two rows are linearly independent.

Spencer's conjecture is backed up by an example (also to be found in [10]) of a matrix A such that $\operatorname{lindisc}(A) = (1 - \frac{1}{n+1}) \operatorname{herdisc}(A)$: Let $m = n+1$. Define $A \in \{0,1\}^{m \times n}$ by

$$a_{ij} = \begin{cases} 1 & \text{if } i = j \text{ or } i = n+1 \\ 0 & \text{else} \end{cases}$$

for all $i \in [m], j \in [n]$. Since A is totally unimodular, it has $\operatorname{herdisc}(A) = 1$.[2] For $p = \frac{1}{m}\mathbf{1}_n$ it is easy to compute $\operatorname{lindisc}(A, p) = 1 - \frac{1}{n+1}$. This also shows

[2] For all $n \in \mathbb{N}$, we write $[n] := \{1, \ldots, n\}$.

that $\operatorname{lindisc}(A) \le 1 - \frac{1}{n+1}$ is the best possible general upper bound for the linear discrepancy of totally unimodular matrices.

There are two results concerning special classes of totally unimodular matrices. Knuth [9] considered the following problem: Given a permutation $\pi \in S_n$ and numbers $x_1, \ldots, x_n \in [0,1]$, find integers $y_1, \ldots, y_n \in \{0,1\}$ such that all 'rounding errors'

$$\left|\sum_{i=1}^{k}(x_i - y_i)\right|, \quad 1 \le k \le n,$$

and

$$\left|\sum_{i=1}^{k}(x_{\pi(i)} - y_{\pi(i)})\right|, \quad 1 \le k \le n,$$

are small. He showed that this problem can be solved with all rounding errors at most $1 - \frac{1}{n+1}$. The proof uses integer flows in networks. As Knuth points out, this problem is a linear discrepancy problem. The corresponding matrix is the incidence matrix of the hypergraph $([n], \{[k], \pi([k]) \mid k \in [n]\})$ and is known to be totally unimodular.

For another class of totally unimodular matrices, Peng and Yan [11] used a combinatorial approach. A matrix A is called *strongly unimodular*, if it is totally unimodular and if each matrix obtained from A by replacing a single non-zero entry by zero is also totally unimodular. Peng and Yan showed that for a strongly unimodular $0,1$ matrix A,

$$\operatorname{lindisc}(A) \le 1 - 3^{-\frac{n+1}{2}}$$

holds. They use a decomposition lemma due to Crama, Loebl and Poljak [2], which states that such a matrix is the union of incidence matrices of digraphs, if all rows contain an even number of ones. In the same paper they gave an upper bound of $1 - \frac{1}{n+1}$ for strongly unimodular $0,1$ matrices which have at most two non-zeros in every row. A simplified proof extending this last result to $-1,0,1$ matrices appeared in [4].

Our Contribution

In this paper we do not follow the approach via the hereditary discrepancy, nor do we use any structure theory for totally unimodular matrices (e.g., Seymour's results [12,13]). Instead, we consider a suitable linear program and apply the theorem of Hoffman and Kruskal. We show

Theorem 1. *For any totally unimodular $m \times n$ matrix A we have*

$$\operatorname{lindisc}(A) \leq 1 - \tfrac{1}{n+1}.$$

If $m \geq 2$, $\operatorname{lindisc}(A) \leq 1 - \frac{1}{m}$ holds.

This result is sharp, as the example given above shows. Our approach via linear programming also yields a characterization of all totally unimodular matrices such that $\operatorname{lindisc}(A) = 1 - \frac{1}{n+1}$:

Theorem 2. *Let A be a totally unimodular $m \times n$ matrix. Then $\operatorname{lindisc}(A) = 1 - \frac{1}{n+1}$ holds if and only if there is either a collection of $n+1$ rows of A such that each n thereof are linearly independent, or $n = 1$ and A contains a single non-zero entry. If $\operatorname{lindisc}(A, p) = 1 - \frac{1}{n+1}$ for some $p \in [0,1]^n$, then $p_i \in \{\frac{1}{n+1}, \ldots, \frac{n}{n+1}\}$ for all $i \in [n]$.*

2. Definitions and Notation

For any number $r \in \mathbb{R}$ write $\lfloor r \rfloor := \max\{z \in \mathbb{Z} | z \leq r\}$ for the largest integer not greater than r, and $\lceil r \rceil := \min\{z \in \mathbb{Z} | z \geq r\}$ for the smallest integer not being less than r. Set $\{r\} := r - \lfloor r \rfloor$, the fractional part of r, and $\operatorname{rd}(r) := \left\lfloor r + \frac{1}{2} \right\rfloor$, the usual rounding of r.

Let $b \in \mathbb{R}^m$. We assume the above notation lifted to vectors in the natural way, e.g., $\lfloor b \rfloor := (\lfloor b_i \rfloor)_{i \in [m]}$. Our proofs are self-contained apart from the well-known theorem of Hoffman and Kruskal [8]:

Theorem (Hoffman, Kruskal). *Let $A \in \mathbb{R}^{m \times n}$ be a totally unimodular matrix. Let $b, b' \in \mathbb{Z}^m$ and $c, c' \in \mathbb{Z}^n$. Then*

$$\{x \in \mathbb{R}^n \,|\, b \leq Ax \leq b', c \leq x \leq c'\}$$

is an integral polyhedron.

Hoffman and Kruskal also showed a converse result: A is totally unimodular if the corresponding polyhedron is integral for all b, b', c, c'. But we shall not need that much understanding of integral polyhedra. For our purposes it suffices to know:

Lemma 1. *Let $P \subseteq \mathbb{R}^n$ be a non-empty integral polyhedron, $f : P \to \mathbb{R}$ a linear mapping and $x \in P$. Then there are $x^-, x^+ \in \mathbb{Z}^n \cap P$ such that $f(x^-) \leq f(x)$ and $f(x^+) \geq f(x)$.*

Thus, if a linear program has an integral polyhedron as its feasible region, optimal solution can be assumed to be integral without loss.

3. Proof of the Linear Discrepancy Bounds

The theorem of Hoffman and Kruskal easily yields $\operatorname{lindisc}(A) < 1$ for a totally unimodular matrix A: Let $p \in [0,1]^n$. The polyhedron

$$P := \{x \in [0,1]^n \mid \lfloor Ap \rfloor \leq Ax \leq \lceil Ap \rceil\}$$

is non-empty, since it trivially contains p. As P is integral, it contains an integral point z. We have $\|A(p-z)\|_\infty < 1$ from the definition of P. In the following proof we improve this approach by adding a suitable objective function.

Proof of Theorem 1. Let $p \in [0,1]^n$ and $b := Ap$. Denote the rows of A by $a_1, \ldots, a_m$. Set $k := \min\{m, n+1\}$. Denoting the usual inner product on $\mathbb{R}^n$ by $\cdot$, we have to show the existence of a $z \in \{0,1\}^n$ such that

$$|b_i - a_i \cdot z| \leq 1 - \tfrac{1}{k}$$

holds for all $i \in [m]$.

Set $I := \{i \in [m] \mid |b_i - \operatorname{rd}(b_i)| < \frac{1}{k}\}$. Let us call these rows *critical* for the moment, because they are the ones where a rounding error $|b_i - a_i \cdot z|$ of more than $1 - \frac{1}{k}$ can be caused by the simple application of the Hoffman-Kruskal theorem above.

We first reduce the problem to the case that there are at most $\min\{n, m\}$ critical rows. Let $I_0 \subseteq I$ be chosen such that a_i, $i \in I_0$, is a basis for the vector space generated by all critical rows. In particular, $|I_0| \leq \min\{m, n\}$.

We now solve the linear discrepancy problem for the matrix consisting of the rows a_i, $i \in ([m] \setminus I) \cup I_0$ only, i.e., we produce a $z \in \{0,1\}^n$ such that $|b_i - a_i \cdot z| < \frac{1}{k}$ holds for all $i \in I_0$ and $|b_i - a_i \cdot z| \leq 1 - \frac{1}{k}$ holds for all $i \in [m] \setminus I$. Let

$$P := \{x \in [0,1]^n \mid \lfloor b \rfloor \leq Ax \leq \lceil b \rceil\}.$$

As A is totally unimodular, P is an integral polyhedron by the theorem of Hoffman-Kruskal. Set $I^- := \{i \in I_0 \mid b_i > \operatorname{rd}(b_i)\}$ and $I^+ := \{i \in I_0 \mid b_i < \operatorname{rd}(b_i)\}$. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \sum_{i \in I^-} (a_i \cdot x - \lfloor b_i \rfloor) + \sum_{i \in I^+} (\lceil b_i \rceil - a_i \cdot x). \tag{1}$$

By definition, f is non-negative on P.

Consider the linear programming problem of minimizing $f(x)$ subject to the constraint $x \in P$. Trivially, p is a feasible solution – i.e., $p \in P$ – and

$$f(p) < |I^- \dot\cup I^+| \, \tfrac{1}{k} \leq 1.$$

As P is integral, there is also an integral solution $z \in \mathbb{Z}^n \cap P$ such that $f(z) \le f(p)$. Thus $f(z) = 0$. Since all parts of the sum in (1) are non-negative, they are all zero. We conclude

$$|a_i \cdot (p - z)| < \tfrac{1}{k} \tag{2}$$

for all $i \in I_0$.

Let $i \in [m] \setminus I$. Then $\{b_i\} \in [\frac{1}{k}, 1 - \frac{1}{k}]$. Since $a_i \cdot z \in \{\lfloor b_i \rfloor, \lceil b_i \rceil\}$ due to $z \in P$, we conclude $|b_i - a_i \cdot z| \le 1 - \frac{1}{k}$.

We end the proof by showing that this z fulfills $\|A(p-z)\|_\infty \le 1 - \frac{1}{k}$, i.e., we also have $|b_i - a_i \cdot z| \le 1 - \frac{1}{k}$ for all $i \in I \setminus I_0$. Assume that $I \setminus I_0 \ne \emptyset$. Then $|I_0| \le k-1$. Let $j \in I \setminus I_0$. As I_0 is a basis for the vector space generated by all critical rows, there are λ_i, $i \in I_0$, such that $a_j = \sum_{i \in I_0} \lambda_i a_i$. Since A is totally unimodular, Cramer's rule implies $\lambda_i \in \{-1, 0, 1\}$ for all $i \in I_0$. Now

$$|a_j \cdot (p - z)| \le \sum_{i \in I_0} |\lambda_i a_i \cdot (p - z)| < \tfrac{1}{k} |I_0| \le 1 - \tfrac{1}{k}$$

by (2). ∎

4. Proof of the Characterization

We finally prove the characterization of those totally unimodular matrices A that have $\mathrm{lindisc}(A) = 1 - \frac{1}{n+1}$.

Proof of Theorem 2. To avoid trivial case distinctions, we leave the case $n = 1$ to the reader and assume $n \ge 2$. Let A be a totally unimodular $m \times n$ matrix such that $\mathrm{lindisc}(A) = 1 - \frac{1}{n+1}$. Let $p \in [0,1]^n$ such that $\mathrm{lindisc}(A, p) = 1 - \frac{1}{n+1}$. Set $b := Ap$ and $I := \{i \in [m] \mid |b_i - \mathrm{rd}(b_i)| \le \frac{1}{n+1}\}$ (note the difference to the set I used in the previous proof). Let I_0 be a subset of I such that a_i, $i \in I_0$, is a basis for the vector space generated by a_i, $i \in I$.

Assume that there is an $i \in I$ such that $\{b_i\} \notin \{\frac{1}{n+1}, 1 - \frac{1}{n+1}\}$. Without loss of generality we may assume $i \in I_0$. By mimicking the proof of Theorem 1 we get a $z \in \{0,1\}^n$ such that $a_i \cdot z = \mathrm{rd}(b_i)$ holds for all $i \in I_0$, $|b_i - a_i \cdot z| \le \sum_{j \in I_0} |b_j - a_j \cdot z| < \frac{n}{n+1}$ holds for all $i \in I \setminus I_0$, and $a_i \cdot z \in \{\lfloor b_i \rfloor, \lceil b_i \rceil\}$ holds for the remaining $i \in [m] \setminus I$. In particular, $\|A(p-z)\|_\infty < \frac{n}{n+1}$ in contradiction to our choice of A and p. We conclude $\{b_i\} \in \{\frac{1}{n+1}, 1 - \frac{1}{n+1}\}$ for all $i \in I$.

From the fact that A is totally unimodular, we know that each a_i, $i \in I \setminus I_0$, can be (uniquely) expressed in the form $a_i = \sum_{j \in I_0} \lambda_{ij} a_j$ with some $\lambda_{ij} \in \{-1, 0, 1\}, j \in I_0$. Let us assume that $|I_0| < n$ or that for each $i \in I \setminus I_0$

at least one of the λ_{ij}, $j \in I_0$, is zero. Again by mimicking the proof of Theorem 1, we find a $z \in \{0,1\}^n$ such that

$$|b_i - a_i \cdot z| \begin{cases} = \frac{1}{n+1} & \text{for all } i \in I_0, \\ \leq \sum_{\substack{j \in I_0 \\ \lambda_{ij} \neq 0}} |b_i - a_i \cdot z| \leq \frac{n-1}{n+1} & \text{for all } i \in I \setminus I_0, \\ < \frac{n}{n+1} & \text{for all } i \in [m] \setminus I. \end{cases}$$

As $n \geq 2$, we have $\|A(p-z)\|_\infty < \frac{n}{n+1}$ contradicting our assumptions. Hence $|I_0| = n$ and there is an $i \in I \setminus I_0$ such that $a_i = \sum_{j \in I_0} \lambda_{ij} a_j$ with all $\lambda_{ij} \in \{-1,1\}, j \in I_0$. In particular, any n of the rows $a_j, j \in I_0 \cup \{i\}$ are linearly independent.

Let A' and b' denote the restrictions of A and b on the rows with index in I_0. As A' has full rank, p is already determined by $A'p = b'$. From $(n+1)b' \in \{1,n\}^n$ we get $(n+1)p \in \mathbb{Z}^n$ (by Cramer's rule the inverse of a totally unimodular matrix is a $(-1,0,1)$ matrix, and thus integral). Clearly, none of the p_i, $i \in [n]$ is 0 or 1 – otherwise we may just put $z_i = p_i$ reducing the dimension of the problem by one. Hence all p_i, $i \in [n]$ are in $\{\frac{1}{n+1}, \dots, \frac{n}{n+1}\}$ as claimed.

Now let A be such that there are $n+1$ rows each n thereof being linearly independent. Without loss of generality we may assume these to be the rows $a_1, \dots, a_{n+1}$. As above there are $\lambda_1, \dots, \lambda_n \in \{-1,1\}$ such that $a_{n+1} = \sum_{i \in [n]} \lambda_i a_i$. Define $b' \in \mathbb{R}^n$ by

$$b'_i := \begin{cases} \frac{1}{n+1} & \text{if } \lambda_i = 1\,, \\ 1 - \frac{1}{n+1} & \text{otherwise,} \end{cases}$$

for all $i \in [n]$. Let A' denote the matrix consisting of the rows $a_1, \dots, a_n$ only. As A' has full rank, the system $A'x = b'$ has a unique solution x. Since $(n+1)b' \in \mathbb{Z}^n$ and A is totally unimodular, $(n+1)x$ is integral. Set $p := \{x\}$ and $b := Ap$. Then $\{b_i\} = \{b'_i\}$ for all $i \in [m]$. We claim that any $z \in \{0,1\}^n$ fulfills $\|A(p-z)\|_\infty \geq 1 - \frac{1}{n+1}$. Let us assume $|a_i \cdot (p-z)| < 1 - \frac{1}{n+1}$ for all $i \in [n]$ (otherwise we are done). From

$$\begin{aligned} \lambda_i a_i \cdot (p - z) &= \lambda_i a_i \cdot x - \lambda_i a_i \cdot \lfloor x \rfloor - \lambda_i a_i \cdot z \\ &= \lambda_i b' - \lambda_i a_i \cdot \lfloor x \rfloor - \lambda_i a_i \cdot z \\ &\in \frac{1}{n+1} + \mathbb{Z} \end{aligned}$$

we conclude $\lambda_a a_i \cdot (p - z) = \frac{1}{n+1}$. Now

$$a_{n+1} \cdot (p - x) = \sum_{i \in [n]} \lambda_i a_i \cdot (p - x) = \frac{n}{n+1}$$

proves the claim. ∎

It is a trivial consequence of the definition of the linear discrepancy that if a matrix B consists of some rows of the matrix A, then $\operatorname{lindisc}(B) \le \operatorname{lindisc}(A)$. In the light of Theorem 2 it makes sense to call a totally unimodular $m \times n$ matrix *critical*, if $m = n+1$ and $\operatorname{lindisc}(A) = 1 - \frac{1}{n+1}$. Theorem 2 then states that a totally unimodular $m \times n$ matrix has linear discrepancy $1 - \frac{1}{n+1}$ if and only if it contains a critical one. The reasoning above also shows that for critical matrices A, there are just two different p such that $\operatorname{lindisc}(A, p) = 1 - \frac{1}{n+1}$ holds, namely the one constructed, call it $p^{(1)}$, and $p^{(2)} := 1 - p^{(1)}$.

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Benjamin Doerr

Mathematisches Seminar II

Christian–Albrechts–Universität zu Kiel

24098 Kiel

Germany

bed@numerik.uni-kiel.de